Achieving control of in-plane elastic waves

M. Brun*, S. Guenneau†
and A.B. Movchan†

*Department of Structural Engineering, University of Cagliari, Cagliari I-09123, Italy

†Department of Mathematical Sciences, Peach Street, Liverpool, L69 3BX, UK

December 4, 2008

Abstract

We derive the elastic properties of a cylindrical cloak for in-plane coupled shear and pressure waves. The cloak is characterized by a rank 4 elasticity tensor with 16 spatially varying entries which are deduced from a geometric transform. Remarkably, the Navier equations retain their form under this transform, which is generally untrue [Milton et al., New J. Phys. **8**, 248 (2006)]. We numerically check that clamped and freely vibrating obstacles located inside the neutral region are cloaked disrespectful of the frequency and the polarization of an incoming elastic wave.

Recently, significant progress has been made on the control of acoustic and electromagnetic waves. Transformation based solutions to the conductivity and Maxwell's equations in curvilinear coordinate systems, subsequently reported by Greenleaf *et al.* [1] and then by Pendry *et al.* [2] and Leonhardt [3], enable one to bend electromagnetic waves around arbitrarily sized and shaped solids. More precisely, the electromagnetic invisibility cloak is a metamaterial which maps a concealment region into a surrounding shell: as a result of the coordinate transformation the permittivity and permeability are strongly heterogeneous and anisotropic within the cloak, yet fulfilling impedance matching with the surrounding vacuum. The cloak thus neither scatter waves nor induces a shadow in the transmitted field. In [4], a cylindrical electromagnetic cloak constructed using specially designed concentric arrays of split ring resonators, was shown to conceal a copper cylinder around

8.5 GHz. The effectiveness of the transformation based cloak was numerically demonstrated solving Maxwell's equations using finite elements for an incident plane wave (far field limit) [5] and for electric line current and magnetic loop sources (near field limit) [6]. In [7], a reduced set of material parameters was introduced to relax the constraint on the permeability, necessarily leading to an impedance mismatch with vacuum which was shown to preserve the cloak effectiveness to a good extent. Other routes to invisibility include reduction of backscatter [8] and cloaking through anomalous localized resonances, the latter one using negative refraction [9, 10]. To date, a plethora of research papers has been published in the fast growing field of transformation optics.

However, transformation based invisibility cloaks applied to certain types of elastodynamic waves in structural mechanics received less attention, since the Navier equations do not usually retain their form under geometric changes [11, 12]. One simplification occurs for cylindrical geometries, whereby out-of-plane shear waves decouple from in-plane waves. However, in-plane shear and pressure waves remain inherently coupled. Earlier proposals for neutral inclusions include using asymptotic and computational methods to find suitable material parameters for coated cylindrical inclusions [13]. The latter has proved successful in the elastostatic context in the case of anti-plane shear and in-plane coupled pressure and shear polarizations. However, neutrality breaks down for finite frequencies.

Other avenues to elastic cloaking should therefore be investigated. For instance, Cummer and Schurig demonstrated that acoustic waves in a fluid undergo the same geometric transform as electromagnetic waves do and therefore retain their form [14]. This result has been since then generalized to three-dimensional acoustic cloaks for pressure waves [16, 15]. Importantly, such cloaks require an anisotropic mass density which can be obtained via a homogenization approach, which presents the advantage to be broadband [17]. Acoustic cloaking for linear surface water waves was chiefly achieved via the same mechanism in between 10 and 15 Hertz [18].

In the present letter, we show that it is also possible to design a cylindrical cloak for in-plane coupled pressure and shear elastic waves. We demonstrate theoretically its unique mechanism and further perform finite element computations checked again analytical calculations of the Green's function for the Navier equations in transformed coordinates. The main difference with previous work [9] is that our elasticity tensor in the transformed coordinates is no longer symmetric, which is a necessary condition for the Navier equa-

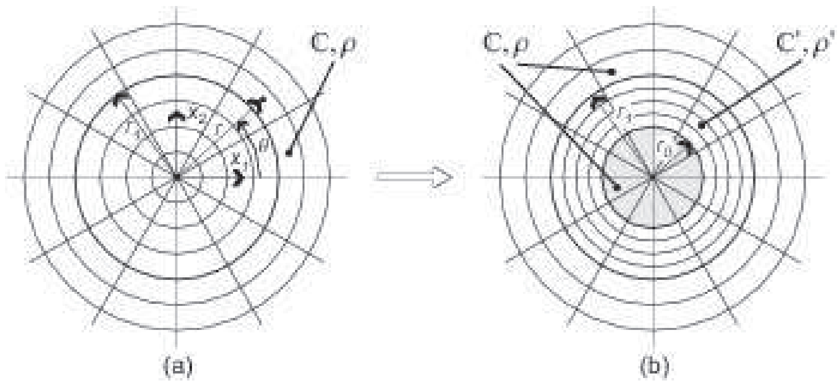

Figure 1: Geometric transform of eqn.(2) from (r, θ) (Fig.1(a)) to (r', θ') (Fig.1(b)); r_0 and r_1 are the inner and the outer radius of the cylindrical cloak, respectively. The elastic constitutive tensor and the density in the undeformed and in the deformed domains are denote by $\mathbb{C}$, ρ and $\mathbb{C}'$, ρ', respectively.

tions to retain their form. Quite remarkably, we find that the density remains a scalar quantity in the transformed coordinates.

We consider the in-plane propagation of time-harmonic elastic waves governed by the Navier equations

$$\nabla \cdot \mathbb{C} : \nabla \mathbf{u} + \rho \omega^2 \mathbf{u} + \mathbf{b} = \mathbf{0} \ , \tag{1}$$

where $\mathbf{u}$ is the displacement, ρ the density, $\mathbb{C}$ the 4^{th}-order constitutive tensor of the linear elastic material and $\mathbf{b} = \mathbf{b}(\mathbf{x})$ represents the spatial distribution of a simple harmonic body force $\hat{\mathbf{b}}(\mathbf{x}, t) = \mathbf{b}(\mathbf{x}) \exp(i\omega t)$, with ω the wave-frequency and t the time.

We introduce the geometric transform $(r, \theta) \rightarrow (r', \theta')$ of [1, 2]

$$\begin{cases} r' = r_0 + \frac{r_1 - r_0}{r_1} r \ , \ \theta' = \theta \ , & \text{for}\, r \leq r1 \\ r' = r \ , \ \theta' = \theta \ , & \text{for}\, r > r1 \end{cases} \tag{2}$$

shown in Fig. (1) and expressed in cylindrical coordinates $r = \sqrt{x_1^2 + x_2^2}$ and $\theta = 2\text{atan}(x_2/(x_1 + \sqrt{x_1^2 + x_2^2}))$, with r_0 and r_1 the inner and outer radii of the circular cloak, respectively.

By application of transformation (2), in the region $r' \in [r_0, r_1]$ the Navier equations (1) are mapped into the equations

$$\nabla \cdot \mathbb{C}' : \nabla \mathbf{u} + \rho' \omega^2 \mathbf{u} = \mathbf{0} \ , \tag{3}$$

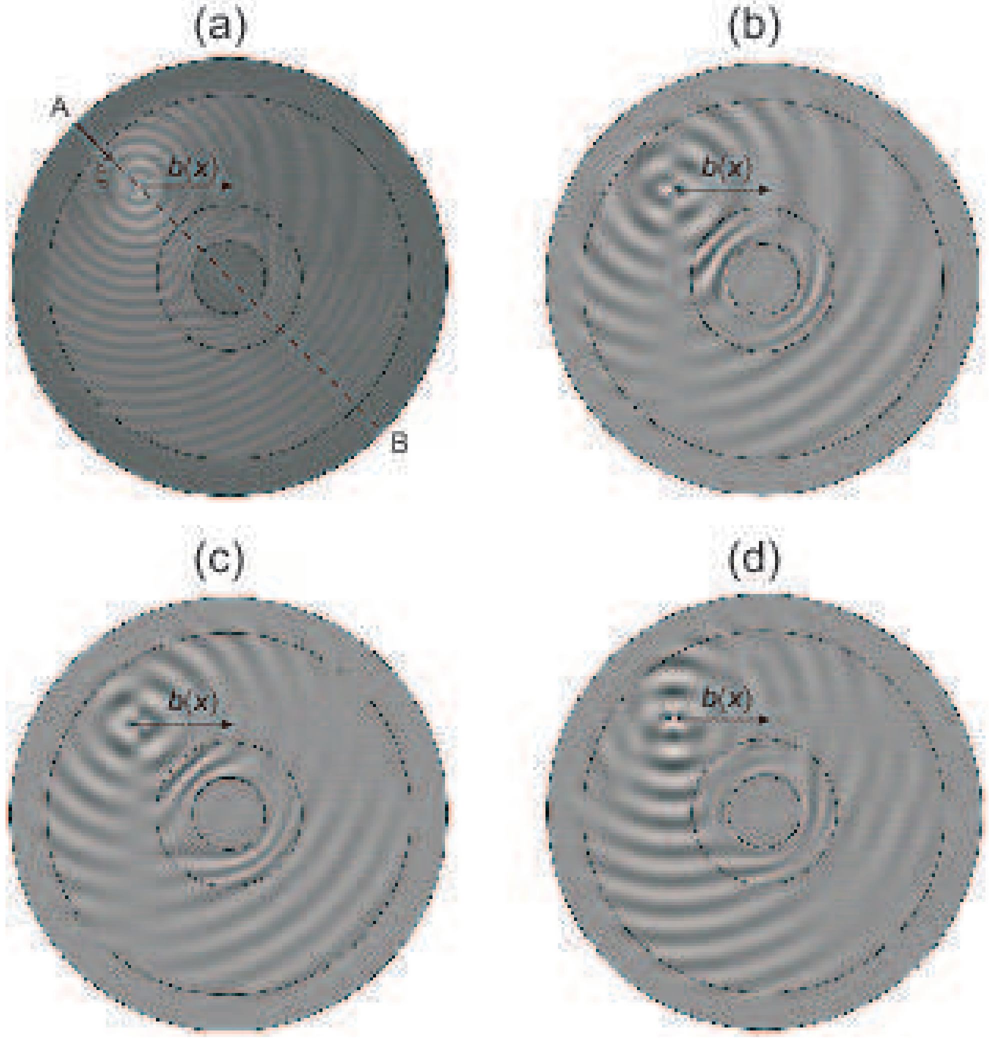

Figure 2: Elastic cloak in an elastic medium subjected to a concentrated load. (a) Displacement magnitude $u = \sqrt{u_1^2 + u_2^2}$; (b) deformation $\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}$; (c) deformation $\varepsilon_{22} = \frac{\partial u_2}{\partial x_2}$; (d) deformation $\varepsilon_{12} = \varepsilon_{21} = \frac{1}{2}(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1})$.

where the body force is assumed to be zero, the stretched density $\rho' =$

$\frac{r-r_0}{r}(\frac{r_1}{r_1-r_0})^2\,\rho$ and elasticity tensor $\mathbb{C}'$ has non zero cylindrical components

$$\begin{array}{ll} \mathbb{C}'_{rrrr}=\frac{r-r_0}{r}(\lambda+2\mu), & \mathbb{C}'_{\theta\theta\theta\theta}=\frac{r}{r-r_0}(\lambda+2\mu), \\ \mathbb{C}'_{rr\theta\theta}=\mathbb{C}'_{\theta\theta rr}=\lambda, & \mathbb{C}'_{r\theta\theta r}=\mathbb{C}'_{\theta r r\theta}=\mu, \\ \mathbb{C}'_{r\theta r\theta}=\frac{r-r_0}{r}\mu, & \mathbb{C}'_{\theta r\theta r}=\frac{r}{r-r_0}\mu, \end{array} \tag{4}$$

with λ and μ the Lamé moduli characterizing the isotropic behavior described by $\mathbb{C}$.

Interestingly, the transformation (2) preserves the isotropy of the density, which remains a scalar (yet spatially varying) quantity in (3), and avoids any coupling between stress and velocity. This is a very unlikely situation for elastodynamic waves propagating in anisotropic heterogeneous media [11]. We also note that the proposed formulation poses no limitations on the applied ω ranging form low to high frequency, as the elasticity tensor does not depend upon ω.

We report the finite element computations performed in the COMSOL multiphysics package. The elastic cloak of equation (4) is embedded in an isotropic elastic material with Lamé moduli $\lambda = 2.3$ and $\mu = 1$ and density $\rho = 1$, which are realistic normalized parameters corresponding to fused silica. The elastic cloak has inner and outer radii $r_0 = 0.2\,m$ and $r_1 = 0.4\,m$, respectively.

The system is excited by an harmonic unit concentrated force applied in direction x_1 and vibrating with angular frequency $\omega = 40$ Hz. A perfectly matched cylindrical layer has been implemented in order to model the infinite elastic medium surrounding the cloak (cf. outer ring on panels a, b, c and d of Fig.2); this has been obtained by application of the geometric transform [19],

$$r'' = r_2 + (1-i)(r-r_2), \qquad \theta'' = \theta, \tag{5}$$

where $r_2 = 1\,m$ is the inner radius of the outer ring in Fig. (2).

In Fig. (2) we clearly see that the wave patterns of the displacement and deformations are smoothly bent around the central region within the cloak (where the magnitudes are nearly zero). Although the coupling of shear and pressure waves generated by the concentrated force creates the optical illusion of interferences, the comparison with the harmonic Green's function in homogeneous elastic space (see [20]) reported in Fig. (3) shows, at least qualitatively, that there is neither forward nor backward scattering. The absence of scattering is better detailed in Fig. 4 where results of Fig. (2) and Fig. (3)

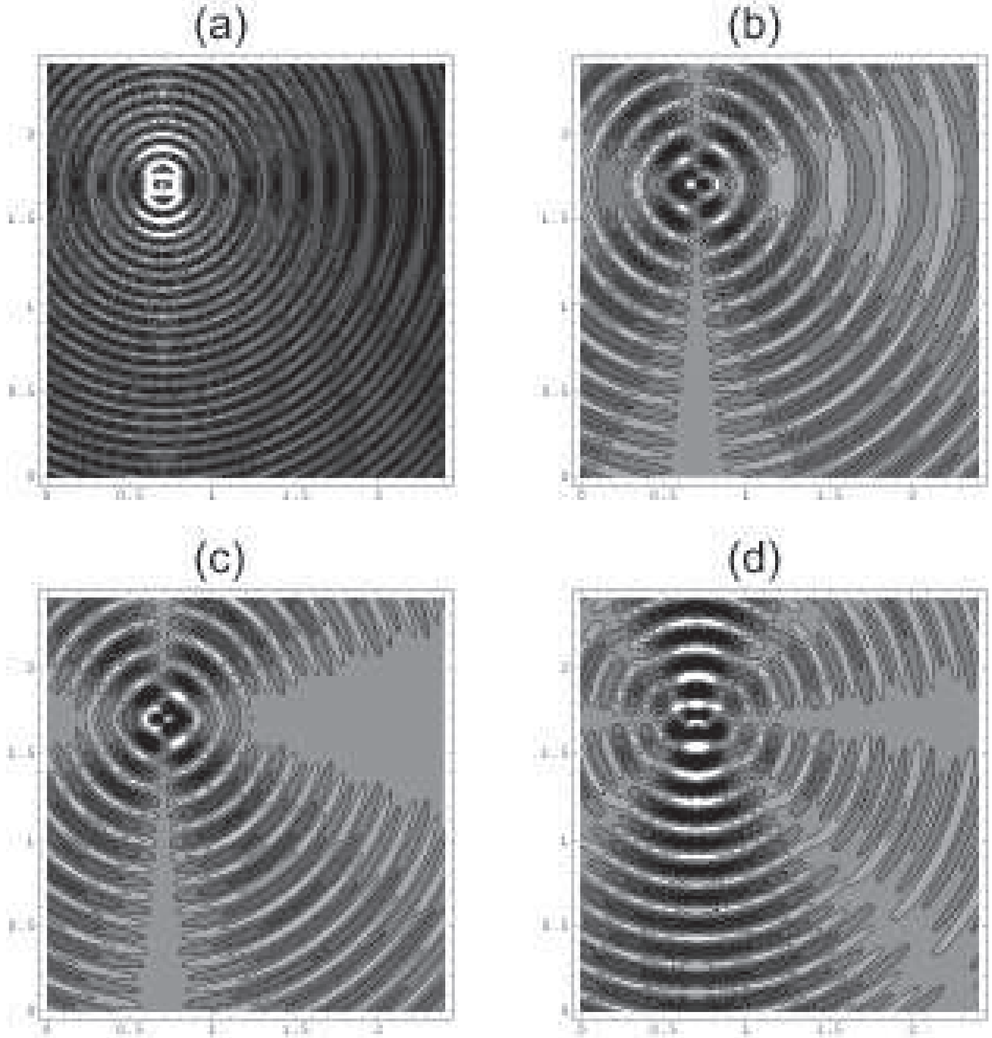

Figure 3: Harmonic Green's function in homogeneous elastic space. (a) Displacement magnitude u; (b) deformation ε_{11}; (c) deformation ε_{22}; (d) deformation ε_{12}.

are compared: the perfect agreement of the displacement fields in the external matrix with and without the cloak is shown, the distortion being bounded to the central region delimited by the cloak. These are non-intuitive results, as the profiles of the horizontal and vertical displacements in Fig. 4 should display a visible phase shift, since the associated acoustic paths are different.

More precisely, let us look at the expression of the elasticity tensor given in (4). On the inner boundary of the cloak, that is for $r = r_0$, its components $\mathbb{C}'_{rrrr}$ and $\mathbb{C}'_{r\theta r\theta}$ vanish, whereas its components $\mathbb{C}'_{\theta\theta\theta\theta}$ and $\mathbb{C}'_{\theta r\theta r}$ tend to infinity. This physically means that pressure and shear waves propagate with an infinite velocity in the θ direction along the inner boundary, which results in a vanishing phase shift between a wave propagating in a homogeneous elastic space and another one propagating around the concealed region: this explains the superimposed profiles of horizontal and vertical displacements in Fig. 4.

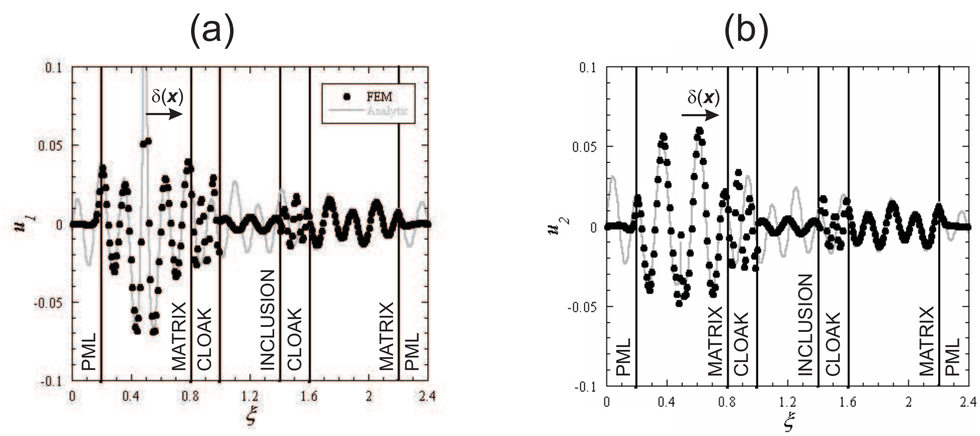

Figure 4: Comparison between numerical results in presence of the elastic cloak of Fig. 2 (black dots) and Green's function in homogeneous elastic space of Fig. 3 (grey lines). Results are given along the line AB detailed in Fig. 2(a). (a) Horizontal displacement u_1; (b) Vertical displacement u_2.

In conclusion, we have proposed an elastic cloak bending the trajectory of in-plane coupled shear and pressure elastic waves around a cylindrical obstacle. The cloak can be designed by the use of heterogeneous density and heterogeneous and anisotropic elastic stiffness; the distribution of the physical properties has been obtained with the introduction of stretched coordinates. Our results open new vistas in cloaking devices for elastodynamic waves in anisotropic media, yet with an isotropic density.

Part of this work was performed while MB was working under the EPSRC grant EP/F027125/1.

References

[1] A. Greenleaf, M. Lassas and G. Uhlmann, *Maths. Res. Lett.* **10**, 1 (2003).

[2] J.B. Pendry, D. Schurig and D.R. Smith, *Science* **312**, 1780 (2006).

[3] U. Leonhardt, *Science* **312**, 1777 (2006).

[4] D. Schurig, J.J. Mock, B.J. Justice, S.A. Cummer, J.B. Pendry, A.F. Starr and D.R. Smith, *Science* **314**, 977 (2006).

[5] S.A. Cummer, B.I. Popa, D. Schurig, D.R. Smith and J.B. Pendry *Phys. Rev. E* **74**, 036621 (2006).

[6] F. Zolla, S. Guenneau, A. Nicolet and J.B. Pendry, *Opt. Lett.* **32**, 1069 (2007).

[7] W. Cai, U.K. Chettiar, A.V. Kildiev and V.M. Shalaev, *Nat. Photonics* **1**, 224 (2007).

[8] A. Alu and N. Engheta, *Phys. Rev. E***72**, 016623 (2005).

[9] G.W. Milton and N.A.P. Nicorovici, *Proc. R. Soc. London Ser. A* **462**, 1364 (2006)

[10] N.A.P. Nicorovici, G.W. Milton, R.C. McPhedran and L.C. Botten, *Opt. Express* **15**, 6314 (2007)

[11] G.W. Milton, M. Briane and J.R. Willis, *New J. Phys.* **8**, 248 (2006).

[12] A. Norris, Proc. Roy. Soc. A **464**, 2411 (2008).

[13] D. Bigoni, S.K. Serkov, M. Valentini and A.B. Movchan, *Int. J. Solids Structures* **35**, 3239 (1998).

[14] S.A. Cummer and D. Schurig *New. J. Phys.* **9**, 45 (2006).

[15] H. Chen and C. T. Chan, *Appl. Phys. Lett.* **91**, 183518 (2007)

[16] S.A. Cummer, B.I. Popa, D. Schurig, D.R. Smith, M. Rahm and J.B. Pendry, *Phys. Rev. Lett* **100**, 024301 (2008).

[17] D. Torrent and J. Sanchez-Dehesa, *New J. Phys.* **10**, 063015 (2008).

[18] M. Farhat, S. Enoch, S. Guenneau and A.B. Movchan, *Phys. Rev. Lett.* **101**, 134501 (2008).

[19] Zheng, Y., Huang, X. *Earth Resources Laboratory 2002 Industry Consortium Meeting*, 1-18, (2002).

[20] Willis, J.R., Movchan, A.B. *J. Mech. Phys. Solids*, **43(3)**, 319, (1995).